AN EXACT RG FORMULATION OF QUANTUM GAUGE THEORY

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A gauge invariant Wilsonian effective action is constructed for pure $SU(N)$ Yang-Mills theory by formulating the corresponding flow equation. Manifestly gauge invariant calculations can be performed *i.e.* without gauge fixing or ghosts. Regularisation is implemented in a novel way which realises a spontaneously broken $SU(N|N)$ supergauge theory. As an example we sketch the computation of the one-loop β function, performed for the first time without any gauge fixing.

1 Introduction and motivation

Our main motivation is to obtain an elegant gauge invariant Wilsonian renormalization group[1] framework formulated directly in the continuum, as a first step for non-perturbative analytic approximation methods.[4,5,6] Quite generally such methods can prove powerful,[a] and of course there is a clear need for a better non-perturbative understanding of gauge theory. However, there are a number of 'spin-offs' in solving this first step: calculations can be made without gauge fixing, continuum low energy gauge invariant Wilsonian effective actions are for the first time precisely defined, a four dimensional gauge invariant 'physical' regulator is discovered, and an intimate link to the Migdal-Makeenko equations[7] is uncovered (which points to a renormalised version of these Dyson-Schwinger equations for Wilson loops[8]).

We refer the reader to the earlier publications for more detailed motivation.[4,5,6] In this lecture we will concentrate on the basic steps and try to keep the discussion straightforward and concrete. The intuition behind these ideas was discussed in an earlier lecture,[4] and all the details may be found in the published papers.[5,6] However in a number of places, especially for the more radical steps, we will try to provide some further intuitive understanding.

In previous exact RG approaches to gauge theory, the authors gauge fixed, and also allowed the effective cutoff to break the gauge invariance. They then sought to recover it in the limit that the cutoff is removed.[9] As we have indicated, the present development follows a very different route. (See also this.[10])

2 The Polchinski equation

We start by casting the established exact RG in a suitable form. We work in D Euclidean dimensions. For two functions $f(\mathbf{x})$ and $g(\mathbf{y})$ and a momentum space kernel $W(p^2/\Lambda^2)$, where Λ is the effective cutoff, we introduce the shorthand:

$$f\cdot W\cdot g := \iint d^Dx\, d^Dy\; f(\mathbf{x})\, W_{\mathbf{xy}}\, g(\mathbf{y}) \quad , \tag{1}$$

[a] See for example the reviews by Bervillier, Tetradis and Wetterich in this volume,and earlier reviews.[2,3]

$$\text{where} \qquad W_{\mathbf{xy}} \equiv \int \frac{d^D p}{(2\pi)^D}\, W(p^2/\Lambda^2)\, \mathrm{e}^{ip\cdot(\mathbf{x}-\mathbf{y})} \quad . \tag{2}$$

Polchinski's[11] version of Wilson's exact RG,[1] for the effective interaction of a scalar field $S^{int}[\varphi]$, may then be written

$$\Lambda\frac{\partial}{\partial\Lambda}S^{int} = -\frac{1}{\Lambda^2}\frac{\delta S^{int}}{\delta\varphi}\cdot c'\cdot\frac{\delta S^{int}}{\delta\varphi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\frac{\delta S^{int}}{\delta\varphi} \quad . \tag{3}$$

Here $c(p^2/\Lambda^2) > 0$ is the effective ultra-violet cutoff, which is implemented by modifying propagators $1/p^2$ to c/p^2. Thus $c(0) = 1$ so that low energies are unaltered, and $c(p^2/\Lambda^2) \to 0$ as $p^2/\Lambda^2 \to \infty$ sufficiently fast that all Feynman diagrams are ultraviolet regulated. We may write the regularised kinetic term (*i.e.* the Gaussian fixed point) as

$$\hat{S} = \tfrac{1}{2}\,\partial_\mu\varphi\cdot c^{-1}\cdot\partial_\mu\varphi \quad . \tag{4}$$

In terms of the total effective action $S[\varphi] = \hat{S} + S^{int}$, and $\Sigma_1 := S - 2\hat{S}$, the exact RG equation reads

$$\Lambda\frac{\partial}{\partial\Lambda}S = -\frac{1}{\Lambda^2}\frac{\delta S}{\delta\varphi}\cdot c'\cdot\frac{\delta\Sigma_1}{\delta\varphi} + \frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\frac{\delta\Sigma_1}{\delta\varphi} \tag{5}$$

up to a vacuum energy term that was discarded in (3).[11] (We have more to say on this below.) The flow in S may be shown directly to correspond to integrating out higher energy modes,[1,2,12,13,14,15] while leaving the partition function $\mathcal{Z} = \int\!\mathcal{D}\varphi\ \mathrm{e}^{-S}$ invariant. (For our purposes we may absorb all source terms into S as spacetime dependent couplings.) We easily see that $\mathcal{Z}$ is invariant if we rewrite (5) as the flow of the measure:

$$\Lambda\frac{\partial}{\partial\Lambda}\,\mathrm{e}^{-S} = -\frac{1}{\Lambda^2}\frac{\delta}{\delta\varphi}\cdot c'\cdot\left(\frac{\delta\Sigma_1}{\delta\varphi}\,\mathrm{e}^{-S}\right) \quad . \tag{6}$$

This leaves the partition function invariant because the right hand side is a total functional derivative.

We are about to generalise these ideas in a novel way so it is as well to settle any nerves about the vacuum energy term we have included in our version of Polchinski's equation (5). Of course Polchinski was safe in discarding this term from the equations as uninteresting. However, his resulting equation then *does not* leave partition function invariant. Rather, it evolves with a scale dependent normalization related to the missing vacuum energy term. The extra term we have included is precisely the one discarded and is precisely the one required to restore the invariance of the partition function. (As a matter of fact, when flowing with respect to a cutoff involving only a subset of these fields the included term can even become anomalous and crucial to the computation of for example β functions.[10])

3 Generalisation to gauge theory

We work with the gauge group $SU(N)$. (All the ideas adapt to other gauge groups.) We write all Lie algebra valued quantities as contracted into the generators. Thus the gauge field appears as $A_\mu(\mathbf{x}) = A^a_\mu(\mathbf{x})\tau^a$, the connection for the covariant derivative $D_\mu = \partial_\mu - iA_\mu$. Often the coupling g is included in D_μ but we can choose to scale it out by absorbing it into A_μ at the expense of a non-standard normalisation for its kinetic term. We will do this for a very important reason, as will become clear shortly.

The generators ${(\tau^a)^i}_j$ are taken to be Hermitian, in the fundamental representation, and orthonormalised as $\mathrm{tr}(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}$. Of course gauge transformations are of the form $\delta A_\mu = D_\mu \cdot \omega := [D_\mu, \omega]$ where $\omega(\mathbf{x}) = \omega^a(\mathbf{x})\tau^a$.

The question then is how to generalise (4,5) so that the flow equation is gauge invariant, whilst leaving the partition function invariant under the flow. It is clear that the regularised kinetic term must now involve the field strength $F_{\mu\nu} := i[D_\mu, D_\nu]$, and some method of covariantizing the cutoff function (which would otherwise break the gauge invariance). Thus we put

$$\hat{S} = \tfrac{1}{2} F_{\mu\nu}\{c^{-1}\}F_{\mu\nu} \quad , \tag{7}$$

where the curly brackets is just a short-hand for any given method of covariantization. To be more explicit about this notation we can write it in terms of the fundamental indices for any two fields u and v in the $N \otimes \bar{N}$ representation:

$$u\{c^{-1}\}v := \int d^Dx\, d^Dy\, u^l_i(\mathbf{x})\, {}_{\mathbf{x}l}^{\;i}\{c^{-1}\}^k_{j\mathbf{y}}\, v^j_k(\mathbf{y}) \quad . \tag{8}$$

Expanding the covariantization in the gauge field A then yields a set of vertices (infinite in number except if c^{-1} is a polynomial):

$$\begin{aligned} u\{c^{-1}\}v = \sum_{m,n=0}^{\infty} \int & d^Dx\, d^Dy\, d^Dx_1 \cdots d^Dx_n\, d^Dy_1 \cdots d^Dy_m \times \\ & \times c^{-1}_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_m}(x_1,\ldots,x_n;y_1,\ldots,y_m;x,y) \times \\ & \times \mathrm{tr}\left[u(x)\, A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\, v(y)\, A_{\nu_1}(y_1)\cdots A_{\nu_m}(y_m)\right]. \end{aligned} \tag{9}$$

Note that in order to keep the notation compact, we label the resulting vertices by the kernel they came from. This procedure can be illustrated diagrammatically as in fig. 1.

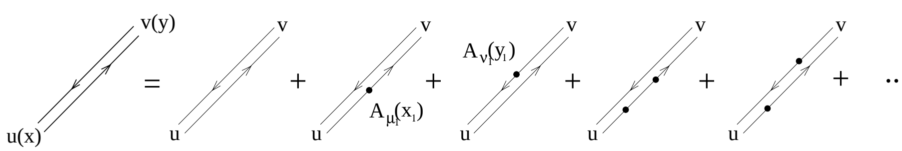

Figure 1. Expansion of the covariantization in terms of gauge fields.

To be concrete about the covariantization, we could insert Wilson lines (as suggested by the diagram):

$$u\{c^{-1}\}v = \iint d^Dx\, d^Dy\;\, c^{-1}_{\mathbf{xy}}\, \mathrm{tr}\, u(\mathbf{x})\Phi[\mathcal{C}_{\mathbf{xy}}]v(\mathbf{y})\Phi^{-1}[\mathcal{C}_{\mathbf{xy}}] \quad , \tag{10}$$

where $c^{-1}_{\mathbf{xy}}$ is defined as in (2), $\mathcal{C}_{\mathbf{xy}}$ is the straight line between $\mathbf{x}$ and $\mathbf{y}$, and the Wilson line is the path ordered exponential:

$$\Phi[\mathcal{C}_{\mathbf{xy}}] = P\exp -i\int_{\mathcal{C}_{\mathbf{xy}}} dz^\mu A_\mu(z) \quad . \tag{11}$$

More generally we could use curved Wilson lines (and a suitable average over their shapes and orientations). Alternatively, we could define the covariantization via the kernels momentum representation:

$$u\{c^{-1}\}v = \mathrm{tr}\int d^D x\, u(\mathbf{x})\, c^{-1}(-D^2/\Lambda^2)\cdot v(\mathbf{x}) \quad . \tag{12}$$

This is the method we actually used for the calculation of the one-loop β function.

Our notation and method of covariantization can be applied to any kernel. It is now easy to completely covariantize Polchinski's equation (5):

$$\Lambda\frac{\partial}{\partial\Lambda}S = -\frac{1}{2\Lambda^2}\frac{\delta S}{\delta A_\mu}\{c'\}\frac{\delta\Sigma_g}{\delta A_\mu} + \frac{1}{2\Lambda^2}\frac{\delta}{\delta A_\mu}\{c'\}\frac{\delta\Sigma_g}{\delta A_\mu} \quad . \tag{13}$$

Here and later the A derivatives are defined contracted into the generators.[4,5] Just as in (5), the first term on the RHS is the *classical* term, yielding the tree corrections, while the second, *quantum*, term, generates the loop corrections.

We have not yet discussed how the coupling g will be incorporated. Recall that we scaled it out of the covariant derivative. It must appear somewhere in (13), and with some thought it turns out that it ends up inside Σ_1, as[5]

$$\Sigma_g = g^2 S - 2\hat{S} \quad . \tag{14}$$

By construction, (13) is manifestly gauge invariant. But we have also been careful not to disturb the structure that guarantees invariance of the partition function: indeed, analogously to (6) we have that the measure still flows by a total derivative:

$$\Lambda\frac{\partial}{\partial\Lambda}\,\mathrm{e}^{-S} = -\frac{1}{\Lambda^2}\frac{\delta}{\delta A_\mu}\{c'\}\left(\frac{\delta\Sigma_g}{\delta A_\mu}\,\mathrm{e}^{-S}\right) \quad . \tag{15}$$

This is the crucial property that ensures that lowering Λ corresponds to integrating out. The only other property required is that momentum integrals really are suppressed above Λ – giving no contribution in the limit $\Lambda \to 0$. If this is the case, since (15) ensures that $\mathcal{Z}$ is unchanged under the flow, the contributions from a given fixed momentum scale must still be in there somewhere, and the only other place they can be, is to already be encoded in the effective action – *i.e.* the modes have been integrated out.

Of course these equations (13,14) are no longer equivalent to the Polchinski equation: the covariantizations in (13) and (7) lead to many new contributions to S already at tree level. Consistency requires these when we come to compute physical quantities and *e.g.* the β function: even when there is no gauge fixing something must generate the eventual contributions normally supplied by ghosts.

The generalised equations (13,14) amount to considering further scale dependent field redefinitions over and above those actually implied by the Polchinski equation.[4,16] One interpretation of our equations is that the flexibility allowed by introducing field redefinitions with each RG 'step' $\Lambda \to \Lambda - \epsilon$, enables us to repair the breaking of the gauge invariance that would otherwise follow from just using the Polchinski equation. A deeper interpretation however is to recognize that the choice of the Polchinski equation is just a convenience and not sacrosanct: there are infinitely many exact RGs just as there are infinitely many ways to block on a lattice.[16]

Whilst (14) shows how the coupling g enters the flow equations, we have not yet related it to the dynamics of the theory. As usual this is achieved through a renormalization condition.

The flow equation is gauge invariant, Lorentz invariant, and may be Taylor expanded in small momenta (for smooth c).[5] The same is true of the solution (provided only that a gauge and Lorentz invariant Taylor expandable initial bare action is chosen). Therefore we know that the lowest non-trivial term in a derivative expansion of the effective action must be $F^2_{\mu\nu}$ up to a so far undetermined coefficient. This can serve to determine g. Thus we write:

$$S = \frac{1}{2g^2(\Lambda)} \,\mathrm{tr}\!\int\! d^4x\, F^2_{\mu\nu} + O(\partial^3/\Lambda) \tag{16}$$

(ignoring the vacuum energy. To clarify, by $O(\partial^3)$ we mean that the other gauge invariant terms, each polynomial in derivatives, would have to contain a part with at least three derivatives; the full gauge invariant term would of course also contain terms with less than three derivatives as required by the covariantization.) Note that in (16), g occupies the position that our bare coupling occupies, but in the effective action we are defining the renormalised coupling and anticipate its running with Λ.

Note a very important consequence of the exact preservation of gauge invariance: A_μ has no wavefunction renormalisation. The proof is so trivial it can take a moment or two to believe it: if the gauge field were to suffer multiplicative wavefunction renormalization by Z, we would have to write $A_\mu \mapsto A_\mu/Z$, destroying the gauge invariance since then $\delta A_\mu = (Z-1)\partial_\mu\omega + D_\mu \cdot \omega$. This argument fails in the gauge fixed theory only because ω is replaced by a ghost field in the BRST transformation leading to pointwise products of fields ($\sim A_\mu\times$ ghost) which are ill defined, and thus the (BRST) invariance is itself ill defined without further renormalization. Our protection mechanism is familiar from the Background Field Method,[17] but we stress that here A_μ is the full quantum field.

Note that had we included the gauge coupling in the covariant derivative as $D_\mu = \partial_\mu - igA_\mu$, A would then run but oppositely to g in such a way that the actual connection gA is fixed. Apart from causing more quantities to run than necessary, it obviously also means that the coefficients of an expansion in g would not be separately gauge invariant. For these reasons, it is very helpful to scale g out in the way that we have done.

The result is that (around the Gaussian fixed point) the exact preservation of gauge invariance ensures that only g could receive divergences; g is the *only* quantity that runs! We see in this observation and (16) some small examples of the power and beauty gained by treating gauge theory in a manifestly gauge invariant manner. The simplest naïve 'textbook' argument about $F^2_{\mu\nu}$ being the only renormalizable (*i.e.* marginal or relevant) interaction is immediately true, and not almost lost in the usually rigorously required complication of ghosts, BRST, Ward-Takahashi, Lee-Zinn-Justin, and Slavnov-Taylor identities.[13]

At this point the sceptic might nevertheless wonder whether gauge fixing is needed in practice. After all, we have only set up the equations. We have not tried to calculate *e.g.* perturbative amplitudes, and it is only at this point that one is forced to gauge fix in the usual approach and indeed other initially gauge invariant approaches such as Migdal-Makeenko equations, and Stochastic quantization.[7,8,13] To see explicitly why no problem arises here, we now sketch how perturbative computations are performed.

4 Perturbation theory

With g^2 scaled out of the action, it counts powers of Plancks constant. From (13) and (14) we see that the classical term is picked out as an order $1/g^2$ piece. Then by iteration we see that S has the usual weak coupling expansion:

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots \quad . \tag{17}$$

Substituting this back into (13,14) and recalling that g can run we find that the beta function takes the expected form

$$\beta := \Lambda \frac{\partial g}{\partial \Lambda} = \beta_1 g^3 + \beta_2 g^5 + \cdots \quad , \tag{18}$$

with of course so far undetermined coefficients. We will see later how they are determined. The flow equation (13) can now be broken up into perturbative pieces:

$$\Lambda \frac{\partial}{\partial \Lambda} S_0 = -\frac{1}{\Lambda^2} \frac{\delta S_0}{\delta A_\mu} \{c'\} \frac{\delta}{\delta A_\mu} (S_0 - 2\hat{S}) \tag{19}$$

$$\Lambda \frac{\partial}{\partial \Lambda} S_1 = 2\beta_1 S_0 - \frac{2}{\Lambda^2} \frac{\delta S_1}{\delta A_\mu} \{c'\} \frac{\delta}{\delta A_\mu} (S_1 - \hat{S}) + \frac{1}{\Lambda^2} \frac{\delta}{\delta A_\mu} \{c'\} \frac{\delta}{\delta A_\mu} (S_0 - 2\hat{S}) \tag{20}$$

and so on.

We can solve these equations by expanding in A. By global $SU(N)$ invariance S expands into traces as illustrated in fig. 2:

$$S = \sum_{n=2}^{\infty} \frac{1}{n} \int d^D x_1 \cdots d^D x_n \, S_{\mu_1 \cdots \mu_n}(\mathbf{x}_1, \cdots, \mathbf{x}_n) \; \mathrm{tr}\, A_{\mu_1}(\mathbf{x}_1) \cdots A_{\mu_n}(\mathbf{x}_n) \; . \tag{21}$$

$\hat{S}$ has the same sort of expansion. Actually, the part of S with four or more gauge fields also has double trace terms and higher powers of traces,[4,5] but to simplify matters we ignore them here.

Expanding S_0 and $\hat{S}$ as in (21) we can see that the two-point vertex satisfies a flow equation diagrammatically of the form of fig. 3. This occurs because the A differentials in (19) open up the traces in fig. 2, attaching the ends of the vertices in fig. 1, as in (9), and thus forming these gauge invariant dumbell shapes. (Note that since the action has no one-point vertices, we must place one gauge field in each

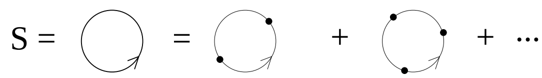

Figure 2. Expansion of the action into traces of gauge fields.

dumbell, making a differentiated two-point vertex. We have not discussed the details of how the A differentials actually act, given that they are also contracted into generators. This requires using the completeness relation and the simple picture we have sketched is not quite correct: extra $O(1/N)$ terms are produced which only contribute to the classical six-point vertices and higher.[5] They will not be needed for this discussion.)

Translating the diagram we have

$$\Lambda\frac{\partial}{\partial\Lambda}S^0_{\mu\nu}(p) = -\frac{1}{2\Lambda^2}c'(p^2/\Lambda^2)\left[S^0_{\mu\lambda}(p) - 2\hat{S}_{\mu\lambda}(p)\right]S^0_{\lambda\nu}(p) + (p_\mu \leftrightarrow -p_\nu) \quad . \tag{22}$$

From expanding (7) we readily obtain all the vertices of $\hat{S}$ in particular

$$\hat{S}_{\mu\nu}(p) = 2\Delta_{\mu\nu}(p)/c(p^2/\Lambda^2)\ , \tag{23}$$

where $\Delta_{\mu\nu}(p) = \delta_{\mu\nu}p^2 - p_\mu p_\nu$ is the usual transverse two-point vertex. By gauge invariance and dimensions, the solution must take a similar form:

$$S^0_{\mu\nu}(p) = 2\Delta_{\mu\nu}(p)/f(p^2/\Lambda^2) \quad , \tag{24}$$

where f is to be determined. From (17), we require $f(0) = 1$ so as to be consistent with (16) in the $g \to 0$ limit. Since (22) is a first order ordinary differential equation this boundary condition determines the solution uniquely. Substituting (23), we readily find this solution to be $f = c$, and thus

$$S^0_{\mu\nu}(p) = \hat{S}_{\mu\nu}(p) \quad . \tag{25}$$

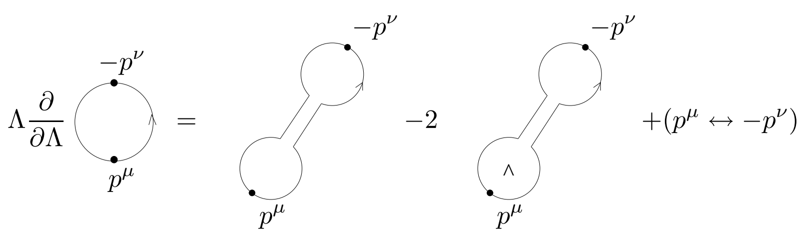

Figure 3. Feynman diagrams for the two-point vertex. Here and later, the empty circle corresponds to S_0 and the circle with a circumflex corresponds to $\hat{S}$.

Higher point vertices fall out even more simply: the fact that the two two-point vertices agree, results in cancellations so that the right hand side of the differential equation contains only lower point vertices that are already determined. This means the result may be immediately integrated. Let us just illustrate with the three-point vertex. After the cancellations the diagrams are those of fig. 4, with solution

$$S^0_{\mu\nu\lambda}(p,q,r) = -\int_\Lambda^\infty \frac{d\Lambda_1}{\Lambda_1^3}\left\{c'_r \hat{S}_{\mu\nu\alpha}(p,q,r)\hat{S}_{\alpha\lambda}(r) + c'_\nu(q;p,r)\hat{S}_{\mu\alpha}(p)\hat{S}_{\alpha\lambda}(r)\right\}$$
$$+\, 2(r_\nu\delta_{\mu\lambda} - r_\mu\delta_{\nu\lambda}) \qquad\qquad + \text{cycles} \quad . \tag{26}$$

Here it should be understood that in the curly brackets we replace Λ with Λ_1, and to the whole expression we add the two cyclic permutations of $(p_\mu, q_\nu, r_\lambda)$. We have taken the continuum limit directly, which is why the top limit is ∞ rather than the overall cutoff Λ_0. The integration constant is just the usual bare three-point vertex as follows from (17) and the $\Lambda \to \infty$ limit of (26) and (16).

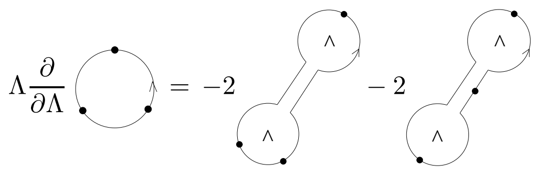

Figure 4. Feynman diagrams for the three-point vertex. The r.h.s. should be summed over cyclic permutations of the momentum labels.

In this way we can continue to solve for higher-point classical vertices and as we will see shortly, also the quantum corrections. *Nota Bene* gauge invariance is maintained at all stages. All the solutions are gauge invariant. (It is a straightforward exercise[4,5] to check that (26) indeed satisfies the naïve Ward identity $p_\mu S^0_{\mu\nu\lambda}(p,q,r) = S^0_{\nu\lambda}(r) - S^0_{\nu\lambda}(q)$.) At no point are we required to invert some kernel that cannot be inverted, which is why gauge fixing is needed in the usual approach (to form the propagator).

Now we can explain how the β function gets determined. We see from (25) that

$$S^0_{\mu\nu}(p) = 2\Delta_{\mu\nu}(p) + O(p^3/\Lambda) \quad . \tag{27}$$

Since $F^2_{\mu\nu}$ is the only gauge invariant construct that gives this, we know that the full classical solution has the form

$$S_0 = \frac{1}{2}\,\mathrm{tr}\int d^4x\, F^2_{\mu\nu} + O(\partial^3/\Lambda) \tag{28}$$

[with interpretation as in (16)]. Comparing this with (16) and (17) we see that –as usual– the classical action *saturates* the renormalisation condition; we learn that S_n must have no $F^2_{\mu\nu}$ component for all loops $n \geq 1$. In particular this means that all the loop corrections to the two-point vertex, *i.e.* $S^n_{\mu\nu}(p)$ $n \geq 1$, must start at $O(p^3)$ or higher.

Now look at the flow equation for $S^1_{\mu\nu}(p)$, which can be seen from (20). The equality (25) kills the middle term on the right hand side. And if we look at only the $O(p^2)$ contribution the left hand side vanishes, by the arguments above. This turns (20) into an algebraic equation and the only way we can solve it is to choose

β_1 precisely to balance (27) against the one-loop term (which by gauge invariance is also proportional to $\Delta_{\mu\nu}$). Thus we have that the $F^2_{\mu\nu}$ part of the one-loop contribution determines β_1:

$$-\frac{1}{4\Lambda^2}\frac{\delta}{\delta A_\mu}\{c'\}\frac{\delta}{\delta A_\mu}(S_0-2\hat{S})\Big|_{F^2_{\mu\nu}} = \beta_1 \quad . \tag{29}$$

The same is true for the higher coefficients and perturbative flow equations and even exactly:

$$\beta(g) \;=\; -\frac{g^3}{4\Lambda^2}\frac{\delta}{\delta A_\mu}\{c'\}\frac{\delta}{\delta A_\mu}(S-2\hat{S})\Big|_{F^2_{\mu\nu}} \quad . \tag{30}$$

5 Wilson loop interpretation

Let us mention that we may draw similar diagrammatic representations to figs. 3,4 for the full flow equation. We can use these to show that the large N limit results in S collapsing to a single trace. (Products of traces decay as $\sim 1/N$ after appropriate changes of variables). Of course these diagrams have a close kinship with 't Hooft's double line notation,[18] but they also have a deeper meaning in terms of fluctuating Wilson loops. This arises because the effective action can be expressed as an 'average' or integral over configurations of Wilson loops (with measure determined by the flow equation) and the covariantized kernels also have an interpretation in terms of integrals over Wilson line pairs [as we have already remarked below (11)]. In this way we may eliminate the gauge field entirely and reexpress the flow equation in terms of the natural low energy order parameter for a gauge theory: the Wilson loop. In the large N limit the whole flow equation collapses to a flow for the measure over the fluctuations of a single Wilson loop. Describing its configuration by a particle going round in a circle we see that we may recast large N $SU(N)$ Yang-Mills as the quantum mechanics of a single particle (with action determined implicitly through the flow equation). The details of these ideas may be found elsewhere.[4,5] Here we keep the discussion firmly concrete, turning to one-loop calculations.

6 $SU(N|N)$ regularisation

From (29), the diagrams contributing to β_1 are those of fig. 5. Here the trace has been broken open in two places and rejoined by the kernel to form two traces.[b] The lower one being empty, just contributes $\mathrm{tr}\,1 = N$.

Once again, it is a simple matter to translate the diagrams to algebra. This time of course we have a momentum integral to do. When we try to compute it however we find the result diverges. This is no surprise: by using covariantized cutoff functions we have effectively implemented covariant higher derivative regularisation, which is known to fail.[19] Actually it cures the superficial divergences of all higher loops except one loop, but of course superficial is not enough. In standard perturbation theory, despite some early controversy[20,21] this problem has

[b] Again we here ignore some terms arising from the completeness relation, which happen to vanish.[5]

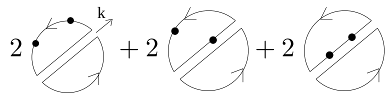

Figure 5. One-loop two-point diagrams, constructed from the four-point vertices of $\Sigma_0 = 2\hat{S} - S_0$.

been cured by supplementing the higher derivative regularisation with a system of Pauli-Villars regulator (PV) fields, the action being bilinear in these fields so that they provide, on integrating out, the missing one loop counterterms.[c][22] This solution turns out to be unwieldy, but worse, here the property of being bilinear in the PV fields is not preserved by the flow:[4] as the gauge field is integrated out higher-point PV interactions are generated.

Instead, we uncovered a system of regulating fields that is more natural from the exact RG point of view.[4,5,6] We have gradually realised that hidden in this formulation are supermatrices and a spontaneously broken local SU($N|N$). Whilst many aspects fell out correctly without us being aware of this structure, the formulation we developed amounts to a unitary gauge in the fermionic directions and is limited to one loop.[4,6] Complete regularisation is achieved in a fully local SU($N|N$) framework as will be seen in the next lectures.[23] (Needless to say, such a framework may be used independently of the Wilsonian RG, and provides a novel and elegant four dimensional 'physical'[d] regularisation for gauge theory.)

Let us stress that there are *two* main threads here. On the one hand we introduce this natural gauge invariant regularisation, as described above. On the other hand, we go on to use it to repair the divergences in the gauge invariant exact RG flow equation, and thus develop a consistent calculational framework in which manifest gauge invariance can be maintained at all stages.

Initially we discovered the regularisation intuitively from the bottom up, introducing interactions in such a way as to guarantee that divergences in one diagram would be cancelled by another at any stage of the flow. Iterating this procedure it turns out that there is very little freedom, and essentially a unique set of Feynman rules for the PV fields is found.[4]

To get a flavour of this, let us just describe how the spectrum of PV fields is determined. First we note that for these high momentum contributions to cancel pairwise amongst diagrams, we need the PV field to have, at least at high momentum, the *same* vertices as A_μ. Therefore we introduce an adjoint field B_μ. In order for it to contribute the opposite sign in one-loop contributions, B must be fermionic. However, when contracted into the Bose-symmetric vertices from A, its anticommutation properties will cause (parts of) the result to vanish. We cure this problem by making it complex, so that we can 'pepper' the A-vertices with B_μ and $\bar{B}_\mu$. However, our problems are not over because we then have many more vertices with B and $\bar{B}$ in, than with just A, leading to many divergent Feynman diagrams

[c]And of course other finite contributions.
[d]In the usual sense that it directly suppresses higher momentum modes.

with no cancelling partner. We can cure this with the following idea. We double the gauge group to $SU(N) \times SU(N)$. The original gauge field will be written A^1_μ. We introduce A^2_μ belonging to the second $SU(N)$. We place $B^{i_1}_{j_2}$ in the 'middle', fundamental with respect to $SU_1(N)$ and complex conjugate fundamental in $SU_2(N)$ (thus oppositely for $\bar{B}^{i_2}_{j_1}$). In this way group theory constrains B to follow $\bar{B}$ and *vice versa*, when tracing round a vertex, restoring the pairwise identification of gauge field diagrams with PV diagrams. Finally, extra divergences arise from the fact that the Bs are massive and thus have longitudinal components. These can be cancelled by introducing bosonic scalars C^i with (covariant) derivative interactions.

We now have a much more elegant way of arriving at this:[6,23] we extend $SU(N)$ Yang-Mills to one built on the supergroup $SU(N|N)$.[24] The gauge field becomes a supergauge field:

$$\mathcal{A}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} \quad , \tag{31}$$

Gauge invariance extends to a full supergauge invariance: $\delta\mathcal{A}_\mu = \nabla_\mu \cdot \omega$, where $\nabla_\mu = \partial_\mu - i\mathcal{A}_\mu$ is the supercovariant derivative and ω is now also a supermatrix. Covariantization of the flow equation under this new group is straightforward. For example, $\hat{S}$ becomes

$$\hat{S} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\{c^{-1}\}\mathcal{F}_{\mu\nu} \quad , \tag{32}$$

where $\mathcal{F}_{\mu\nu}$ is the superfield strength, covariantization in (9) is via $\mathcal{A}$ and, most importantly, all traces here and elsewhere are replaced by supertraces. This last step is necessary to preserve the property of cyclicity when using supermatrices (which in turn is necessary to prove gauge invariance). The supertrace is defined by

$$\mathrm{str}\,\mathbf{X} = \mathrm{str}\begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = \mathrm{tr}\,X^{11} - \mathrm{tr}\,X^{22} \quad . \tag{33}$$

This means that in the quantum corrections (such as the β_1 calculation above) the $\mathrm{tr}\,1 = N$ parts are now replaced by $\mathrm{str}\,1 = 0$! The symmetry between bosonic and fermionic contributions causes quantum corrections to vanish, just as it can with normal space-time supersymmetry. (Here however the supersymmetry is implemented in a novel way: on the fibre.)

We are not interested in such a complete cancellation but then we are not interested in massless fermionic vector fields B_μ either. We must make these fields massive without destroying the cancellation properties at high energies (which will then act as a regulator) and without disturbing the original $SU(N)$ gauge invariance. Fortunately we know how to do this: we introduce a superscalar 'Higgs' to spontaneously break the theory along just the fermionic directions, *i.e.*

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix} \qquad \text{such that} \qquad <\mathcal{C}>= \Lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{34}$$

In unitary gauge B eats D and becomes massive of order Λ, leaving behind the 'physical' Higgs

$$\begin{pmatrix} C^1 & 0 \\ 0 & C^2 \end{pmatrix} \quad ,$$

whose mass is also (or may naturally be chosen to be) of order Λ, and an undisturbed $SU(N)\times SU(N)$ gauge invariance. Remarkably, this construction leads to the *same* spectrum and interactions as the bottom up approach!

There are a number of subtleties that we have skated over: we need to higher-derivative regularise also the Higgs sector and this introduces another cutoff function, which here for simplicity we have ignored.[6,23] Of course A^1 remains massless, but the second gauge field A^2 also remains massless, and is unphysical since it has a wrong sign action[4] (a consequence of the supertrace[6]). This leads to a source of unitarity breaking which however disappears in the limit the cutoff is removed.[23] (In the exact RG approach this corresponds to setting $\Lambda_0 \to \infty$.) This is because A^1 and A^2 are the gauge fields of the direct product subgroup $SU(N) \times SU(N)$, and are therefore charge neutral under each others gauge group, thus at energies much less than Λ_0, we are left only with these gauge fields which decouple. To prove this it is only necessary to show that the amplitudes that mix the two $SU(N)$s all vanish at fixed momentum as $\Lambda_0 \to \infty$. This does indeed follow, from gauge invariance and dimensional considerations.[23] Finally, while quantum corrections are now finite, this results from cancellation of separately divergent pieces and thus care is needed in defining these conditionally convergent integrals (*e.g.* by employing a gauge invariant preregularisation[4,5,6]).

Recomputing the one loop β function with the manifestly gauge invariant exact RG extended to this realisation of spontaneously broken $SU(N|N)$, we find that the momentum integral is now finite and furthermore the integrand is a total divergence. The resulting surface integral is independent of the choice of cutoff function $c(p^2/\Lambda^2)$ and treating the implied limits carefully yields the famous result[6]

$$\beta_1 = -\frac{11}{3}\frac{N}{(4\pi)^2} \quad ,$$

thus furnishing the first calculation of the one-loop β function without fixing the gauge.

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